Geometry of Quaternionic Kähler connections with torsion

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Abstract

The target space of a (4,0) supersymmetric two-dimensional sigma model with Wess-Zumino term has a connection with totally skew-symmetric torsion and holonomy contained in Sp(n).Sp(1), QKT-connection. We study the geometry of QKT-connections. We find conditions to the existence of a QKT-connection and prove that if it exists it is unique. Studying conformal transformations we obtain a lot of (compact) examples of QKT manifolds. We present a (local) description of 4-dimensional homogeneous QKT structures relying on the known result of naturally reductive homogeneous Riemannian manifolds. We consider Einstein-like QKT manifold and find closed relations with Einstein-Weyl geometry in dimension four.

Running title: Quaternionic Kähler with torsion

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1 Introduction and statement of the results

An almost hyper complex structure on a 4n-dimensional manifold M is a triple $H = (J_\alpha), \alpha = 1, 2, 3,$ of almost complex structures $J_\alpha : TM \to TM$ satisfying the quaternionic identities $J_\alpha^2 = -id$ and $J_1J_2 = -J_2J_1 = J_3$. When each J_α is a complex structure, H is said to be a hyper complex structure on M.

An almost quaternionic structure on M is a rank-3 subbundle $Q \subset End(TM)$ which is locally spanned by almost hypercomplex structure $H = (J_\alpha)$; such a locally defined triple H will be called an admissible basis of Q. A linear connection ∇ on TM is called quaternionic connection if ∇ preserves Q, i.e. $\nabla_X\sigma \in \Gamma(Q)$ for all vector fields X and smooth sections $\sigma \in \Gamma(Q)$. An almost

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quaternionic structure is said to be a quaternionic if there is a torsion-free quaternionic connection. A Q-hermitian metric is a Riemannian metric which is Hermitian with respect to each almost complex structure in Q. An almost quaternionic (resp. quaternionic) manifold with Q-hermitian metric is called an almost quaternionic Hermitian (resp. quaternionic hermitian) manifold

For $n = 1$ an almost quaternionic structure is the same as an oriented conformal structure and it turns out to be always quaternionic. When $n \geq 2$, the existence of torsion-free quaternionic connection is a strong condition which is equivalent to the 1-integrability of the associated GL(n,H)SP(1) structure [10, 32, 42]. If the Levi-Civita connection of a quaternionic hermitian manifold (M, g, Q) is a quaternionic connection then (M, g, Q) is called Quaternionic Kähler (briefly QK). This condition is equivalent to the statement that the holonomy group of g is contained in SP(n).SP(1) [1, 2, 39, 40, 25]. If on a QK manifold there exist an admissible basis (H) such that each almost complex structure $(J_\alpha) \in (H), \alpha = 1, 2, 3$ is parallel with respect to the Levi-Civita connection then the manifold is called hyper Kähler (briefly HK). In this case the holonomy group of g is contained in SP(n).

The notions of quaternionic manifolds arise in a natural way from the theory of supersymmetric sigma models. The geometry of the target space of two-dimensional sigma models with extended supersymmetry is described by the properties of a metric connection with torsion [14, 22]. The geometry of (4,0) supersymmetric two-dimensional sigma models without Wess-Zumino term (torsion) is a hyper Kähler manifold. In the presence of torsion the geometry of the target space becomes hyper Kähler with torsion (briefly HKT) [23]. This means that the complex structures $J_\alpha, \alpha = 1, 2, 3,$ are parallel with respect to a metric quaternionic connection with totally skew-symmetric torsion [23]. Local (4,0) supersymmetry requires that the target space of two dimensional sigma models with Wess-Zumino term be either HKT or quaternionic Kähler with torsion (briefly QKT) [31] which means that the quaternionic subbundle is parallel with respect to a metric linear connection with totally skew-symmetric torsion and the torsion 3-form is of type (1,2)+(2,1) with respect to all almost complex structures in Q. The target space of two-dimensional (4,0) supersymmetric sigma models with torsion coupled to (4,0) supergravity is a QKT manifold [24]. If the torsion of a QKT manifold is a closed 3-form then it is called strong QKT manifold. The properties of HKT and QKT geometries strongly resemble those of HK and QK ones, respectively. In particular, HKT [23] and QKT [24] manifolds admit twistor constructions with twistor spaces which have similar properties to those of HK [21] and QK [39, 40, 41].

The main object of interest in this article is the differential geometric properties of QKT manifolds. We find necessary and sufficient conditions to the existence of a QKT connection in terms of the Kähler 2-forms and show that the QKT-connection is unique if dimension is at least 8 (see Theorem 2.2 below). We prove that the QKT manifolds are invariant under conformal transformations of the metric. This allows us to present a lot of (compact) examples of QKT manifolds. In particular, we show that the compact quaternionic Hopf manifolds studied in [34], which do not admit a QK structure, are QKT manifolds. In the compact case we show the existence of Gauduchon metric i.e. the unique conformally equivalent QKT structure with co-closed torsion 1-form.

It is shown in [24] that the twistor space of a QKT manifold is always complex manifold provided the dimension is at least 8. It admits complex contact (resp. Kähler) structure if the torsion 4-form is of type (2,2) and some additional nondegeneratity (positivity) conditions are fulfilled [24]. Most of the known examples of QKT manifolds are homogeneous constructed in[33]. However, there are no homogeneous proper QKT manifolds (i.e. QKT which is not QK or HKT) with torsion 4-form of type (2,2) in dimensions greater than four by the result of [33]. We generalise this result showing that there are no proper QKT manifolds with torsion 4-form of type (2,2) provided that the torsion

is parallel and dimension is at least 8.

In dimension 4 a lot of examples of QKT manifolds are known [24, 33]. In particular, examples of homogeneous QKT manifolds are constructed in [33]. We notice that there are many (even strong) QKT structures in dimension 4, all depending on an arbitrary 1-form. We give a local description of 4-dimensional QKT manifolds with parallel torsion; namely such a QKT manifold is a Riemannian product of a real line and a 3-dimensional Riemannian manifold. We observe that homogeneous QKT manifolds are precisely naturally reductive homogeneous Riemannian manifolds, the objects which are well known. We present a complete local description (up to an isometry) of 4-dimensional homogeneous QKT which was known in the setting of naturally reductive homogeneous 4-manifold [27]. In the last section we consider 4-dimensional Einstein-like QKT manifold and find a closed relation with Einstein-Weyl geometry in dimension four. In particular, we show that every 4-dimensional HKT manifold is of this type.

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2 Characterisations of QKT connection

Let $(M, g, (J_\alpha) \in Q, \alpha = 1, 2, 3)$ be a 4n-dimensional almost quaternionic manifold with Q-hermitian Riemannian metric g and an admissible basis (J_α). The Kähler form F_α of each J_α is defined by $F_\alpha = g(., J\alpha.)$. The corresponding Lee forms are given by $\theta_\alpha = \delta F_\alpha \circ J_\alpha$.

For an r-form ψ we denote by $J_\alpha \psi$ the r-form defined by
$J_\alpha \psi(X_1, ..., X_r) := (-1)^r \psi(J_\alpha X_1, ..., J_\alpha X_r), \alpha = 1, 2, 3$. Then $(d^c\psi)_\alpha = (-1)^r J_\alpha d\psi$. We shall use the notations $d_\alpha F_\beta := (d^c F_\beta)_\alpha$, i.e. $d_\alpha F_\beta(X, Y, Z) = -dF_\beta(J_\alpha X, J_\alpha Y, J_\alpha Z), \alpha, \beta = 1, 2, 3$.

We recall the decomposition of a skew-symmetric tensor $P \in \Lambda^2 T^*M \otimes TM$ with respect to a given almost complex structure J_α. The (1,1), (2,0) and (0,2) part of P are defined by $P^{1,1}(J_\alpha X, J_\alpha Y) = P^{1,1}(X, Y), P^{2,0}(J_\alpha X, Y) = J_\alpha P^{2,0}(X, Y), P^{0,2}(J_\alpha X, Y) = -J_\alpha P^{0,2}(X, Y)$, respectively.

For each $\alpha = 1, 2, 3$, we denote by dF^+_α (resp. dF^-_α) the $(1, 2) + (2, 1)$-part (resp. $(3, 0) + (0, 3)$-part) of dF_α with respect to the almost complex structure J_α. We consider the following 1-forms

$$\theta_{\alpha,\beta} = -\frac{1}{2} \sum_{i=1}^{4n} dF^+_\alpha(X, e_i, J_\beta e_i), \quad \alpha, \beta = 1, 2, 3.$$

Here and further $e_1, e_2, \ldots, 4n$ is an orthonormal basis of the tangential space.

Note that $\theta_{\alpha,\alpha} = \theta_\alpha$.

The Nijenhuis tensor N_α of an almost complex structure J_α is given by
$N_\alpha(X, Y) = [J_\alpha X, J_\alpha Y] - [X, Y] - J_\alpha[J_\alpha X, Y] - J_\alpha[X, J_\alpha Y]$.

The celebrated Newlander-Nirenberg theorem [30] states that an almost complex structure is a complex structure if and only if its Nijenhuis tensor vanishes.

Let ∇ be a quaternionic connection i.e.

$$\nabla J_\alpha = -\omega_\beta \otimes J_\gamma + \omega_\gamma \otimes J_\beta, \tag{2.1}$$

where the $\omega_\alpha, \alpha = 1, 2, 3$ are 1-forms.

Here and henceforth (α, β, γ) is a cyclic permutation of $(1, 2, 3)$.

Let $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ be the torsion tensor of type (1,2) of ∇. We denote by the same letter the torsion tensor of type (0,3) given by $T(X,Y,Z) = g(T(X,Y),Z)$. The Nijenhuis tensor is expressed in terms of ∇ as follows

$$
\begin{aligned}
(2.2)\qquad N_\alpha(X,Y) &= 4T_\alpha^{0,2}(X,Y) \\
&+ (\nabla_{J_\alpha X} J_\alpha)(Y) - (\nabla_{J_\alpha Y} J_\alpha)(X) - (\nabla_Y J_\alpha)(J_\alpha X) + (\nabla_X J_\alpha)(J_\alpha Y),
\end{aligned}
$$

where the (0,2)-part $T_\alpha^{0,2}$ of the torsion with respect to J_α is given by

$$
(2.3)\qquad T_\alpha^{0,2}(X,Y) = \frac{1}{4}\left(T(X,Y) - T(J_\alpha X, J_\alpha Y) + J_\alpha T(J_\alpha X, Y) + J_\alpha T(X, J_\alpha Y)\right).
$$

We recall that if a 3-form ψ is of type (1,2)+(2,1) with respect to an almost complex structure J then it satisfies the equality

$$
(2.4)\qquad \psi(X,Y,Z) = \psi(JX,JY,Z) + \psi(JX,Y,JZ) + \psi(X,JY,JZ).
$$

Definition. An almost quaternionic hermitian manifold $(M, g, (H_\alpha) \in Q)$ is *QKT manifold* if it admits a metric quaternionic connection ∇ with totally skew symmetric torsion which is (1,2)+(2,1)-form with respect to each $J_\alpha, \alpha = 1,2,3$. If the torsion 3-form is closed then the manifold is said to be *strong QKT manifold*.

It follows that the holonomy group of ∇ is a subgroup of SP(n).SP(1).

By means of (2.1), (2.2) and (2.4), the Nijenhuis tensor N_α of $J_\alpha, \alpha = 1,2,3$, on a QKT manifold is given by

$$
(2.5)\qquad N_\alpha(X,Y) = A_\alpha(Y) J_\beta X - A_\alpha(X) J_\beta Y - J_\alpha A_\alpha(Y) J_\gamma X + J_\alpha A_\alpha(X) J_\gamma Y,
$$

where

$$
(2.6)\qquad A_\alpha = \omega_\beta + J_\alpha \omega_\gamma.
$$

Remark 1. The definition of QKT manifolds given above is equivalent to that given in [24] because the requirement the torsion to be (1,2)+(2,1)-form with respect to each $J_\alpha, \alpha = 1,2,3$, is equivalent, by means of (2.5), to the fourth condition of (4) in [24]. The torsion of ∇ is (1,2)+(2,1)-form with respect to any (local) almost complex structure $J \in Q$ [24]. This follows also from (2.5) and the general formula (6) in [4] which expresses N_J in terms of $N_{J_1}, N_{J_2}, N_{J_3}$. In fact, it is sufficient that the torsion is a (1,2)+(2,1)-form with respect to the only two almost complex structures of (H) since the formula (3.4.4) in [3]) gives the necessary expression of N_{J_3} by N_{J_1} and N_{J_2}. Indeed, it is easy to see that the formula (3.4.4) in [3] holds for the (0,2)-part $T_\alpha^{0,2}, \alpha = 1,2,3$, of the torsion. Hence, the vanishing of the (0,2)-part of the torsion with respect to any two almost complex structures in (H) implies the vanishing of the (0,2)-part of T with respect to the third one.

On a QKT manifold there are three naturally associated 1-forms to the torsion defined by

$$
(2.7)\qquad t_\alpha(X) = -\frac{1}{2}\sum_{i=1}^{4n} T(X, e_i, J_\alpha e_i), \quad \alpha = 1,2,3.
$$

We have

Proposition 2.1 *On a QKT manifold* $J_1 t_1 = J_2 t_2 = J_3 t_3$.

Proof. Applying (2.4) with respect to J_β we obtain

$$\begin{aligned} t_\alpha(X) &= -\frac{1}{2}\sum_{i=1}^{4n} T(X, e_i, J_\alpha e_i) = -\frac{1}{2}\sum_{i=1}^{4n} T(X, J_\beta e_i, J_\gamma e_i) \\ &= \frac{1}{2}\sum_{i=1}^{4n} T(J_\beta X, e_i, J_\gamma e_i) - \frac{1}{2}\sum_{i=1}^{4n} T(J_\beta X, J_\beta e_i, J_\alpha e_i) + \frac{1}{2}\sum_{i=1}^{4n} T(X, e_i, J_\alpha e_i). \end{aligned}$$

The last equality implies $t_\alpha = J_\beta t_\gamma$ which proves the assertion. **Q.E.D.**

The 1-form $t = J_\alpha t_\alpha$ is independent of the chosen almost complex structure J_α by Proposition 2.1. We shall call it *the torsion 1-form* of a given QKT manifold.

Remark 2. Every QKT manifold is a quaternionic manifold. This is an immediate consequence of (2.5) and Proposition 2.3 in [4].

However, the converse to the above property is not always true. In fact, we have

Theorem 2.2 *Let $(M, g, (J_\alpha \in Q)$ be a 4n-dimensional ($n > 1$) quaternionic manifold with Q-hermitian metric g. Then M admits a QKT structure if and only if the following conditions hold*

$$(d_\alpha F_\alpha)^+ - (d_\beta F_\beta)^+ = \frac{1}{2}\left(K_\alpha \wedge F_\beta - J_\beta K_\beta \wedge F_\alpha - (K_\beta - J_\alpha K_\alpha) \wedge F_\gamma\right), \tag{2.8}$$

where $(d_\alpha F_\alpha)^+$ denotes the (1,2)+(2,1) part of $(d_\alpha F_\alpha)$ with respect to the $J_\alpha, \alpha = 1, 2, 3$. The 1-forms $K_\alpha, \alpha = 1, 2, 3$, are given by

$$K_\alpha = \frac{1}{1-n}\left(J_\beta \theta_\alpha + \theta_{\alpha,\gamma}\right). \tag{2.9}$$

The metric quaternionic connection ∇ with torsion 3-form of type (1,2)+(2,1) is unique and is determined by

$$\nabla = \nabla^g + \frac{1}{2}\left((d_\alpha F_\alpha)^+ - \frac{1}{2}\left(J_\alpha K_\alpha \wedge F_\gamma + K_\alpha \wedge F_\beta\right)\right), \tag{2.10}$$

where ∇^g is the Levi-Civita connection of g.

Proof. To prove the 'if' part, let ∇ be a metric quaternionic connection satisfying (2.1) which torsion T has the required properties. We follow the scheme in [17]. Since T is skew-symmetric we have

$$\nabla = \nabla^g + \frac{1}{2}T. \tag{2.11}$$

We obtain using (2.1) and (2.11) that

$$\begin{aligned} \frac{1}{2}\left(T(X, J_\alpha Y, Z) + (T(X, Y, J_\alpha Z)\right) &= -g\left((\nabla^g_X J_\alpha)Y, Z\right) \\ &+ \omega_\beta(X)F_\gamma(Y,Z) - \omega_\gamma(X)F_\beta(Y,Z). \end{aligned} \tag{2.12}$$

The tensor $\nabla^g J_\alpha$ is decomposed by parts according to $\nabla J_\alpha = (\nabla J_\alpha)^{2,0} + (\nabla J_\alpha)^{0,2}$, where [17]

$$g\left((\nabla^g_X J_\alpha)^{2,0} Y, Z\right) = \frac{1}{2}\left((dF_\alpha)^+(X, J_\alpha Y, J_\alpha Z) - (dF_\alpha)^+(X,Y,Z)\right) \tag{2.13}$$

$$g\left((\nabla^g_X J_\alpha)^{0,2} Y, Z\right) = \frac{1}{2}\left(g(N_\alpha(X,Y), J_\alpha Z) - g(N_\alpha(X,Z), J_\alpha Y) - g(N_\alpha(Y,Z), J_\alpha X)\right) \tag{2.14}$$

Taking the (2,0) part in (2.12) we obtain using (2.13) that

$$
\begin{aligned}
T(X,J_\alpha Y,Z)+T(X,Y,J_\alpha Y) &= (dF^+_\alpha(X,J_\alpha Y,J_\alpha Z)-(dF^+_\alpha(X,Y,Z)\\
&+ C_\alpha(X)F_\gamma(Y,Z)+C_\alpha(J_\alpha X)F_\beta(Y,Z),
\end{aligned}
\tag{2.15}
$$

where

$$
C_\alpha=\omega_\beta-J_\alpha\omega_\gamma. \tag{2.16}
$$

The cyclic sum of (2.15) and the fact that T and $(dF_\alpha)^+$ are (1,2)+(2,1)-forms with respect to each J_α, gives

$$
T=(d_\alpha F_\alpha)^+-\frac{1}{2}\left(J_\alpha C_\alpha\wedge F_\gamma+C_\alpha\wedge F_\beta\right). \tag{2.17}
$$

Further, we take the contractions in (2.17) to get

$$
\begin{aligned}
J_\alpha t_\alpha &= -\theta_\alpha-J_\beta C_\alpha,\\
J_\alpha t_\alpha &= -J_\gamma\theta_{\beta,\alpha}-nJ_\gamma C_\beta,\\
J_\alpha t_\alpha &= J_\beta\theta_{\gamma,\alpha}-nJ_\alpha C_\gamma
\end{aligned}
\tag{2.18}
$$

Using Proposition 2.1, (2.6) and (2.16), we obtain consequently from (2.18) that

$$
A_\alpha=J_\alpha C_\beta+J_\gamma C_\gamma=J_\beta\left(\theta_\gamma-\theta_\beta\right), \tag{2.19}
$$

$$
(n-1)J_\beta C_\alpha=\theta_\alpha-J_\beta\theta_{\alpha,\gamma}. \tag{2.20}
$$

Then (2.8) and (2.9) follow from (2.17) and (2.20).

For the converse, we define ∇ by (2.10). To complete the proof we have to show that ∇ is a quaternionic connection. We calculate

$$
\begin{aligned}
g\left((\nabla_X J_\alpha)Y,Z\right) &= g\left((\nabla^g_X J_\alpha)Y,Z\right)+\frac{1}{2}\left(T(X,J_\alpha Y,Z)+T(X,Y,J_\alpha Z)\right)\\
&= \omega_\beta(X)F_\gamma(Y,Z)-\omega_\gamma(X)F_\beta(Y,Z),
\end{aligned}
$$

where we used (2.13), (2.14), (2.19), (2.9), (2.6), (2.16) and the compatibility condition (2.8) to get the last equality. The uniqueness of ∇ follows from (2.10) as well as from Theorem 10.3 in [32] which states that any quaternionic connection is entirely determined by its torsion (see also [18]). **Q.E.D.**

In the case of HKT manifold, $K_\alpha=dF^-_\alpha=0$ and Theorem 2.2 is a consequence of the general results in [17] (see also [20]) which imply that on a hermitian manifold there exists a unique linear connection with totally skew-symmetric torsion preserving the metric and the complex structure, the Bismut connection. This connection was used by Bismut [9] to prove a local index theorem for the Dolbeault operator on non-Kähler manifold. The geometry of this connection is referred to KT-geometry by physicists. Obstructions to the existence of (non-trivial) Dolbeault cohomology groups on a compact KT-manifold are presented in [5].

We note that (2.19) and (2.20) are also valid in the case $n=1$.

We get, as a consequence of the proof of Theorem 2.2, the following integrability criterion which is discovered in dimension 4 in [19].

Proposition 2.3 *The Nijenhuis tensors of a QKT manifold depend only on the difference between the Lie forms. In particular, the almost complex structures J_α on a QKT manifold $(M,(J_\alpha)\in Q,g,\nabla)$ are integrable if and only if*

$$
\theta_\alpha=\theta_\beta=\theta_\gamma
$$

Proof. The Nijenhuis tensors are given by (2.5) and (2.19). **Q.E.D.**

Corollary 2.4 *On a 4n-dimensional QKT manifold the following formulas hold*

$$J_\beta\theta_{\alpha,\gamma} = -J_\gamma\theta_{\alpha,\beta},$$

$$(n^2+n)\theta_\alpha - n\theta_\beta - n^2\theta_\gamma + J_\gamma\theta_{\beta,\alpha} + nJ_\alpha\theta_{\gamma,\beta} - (n+1)J_\beta\theta_{\alpha,\gamma} = 0. \tag{2.21}$$

If $n = 1$ *then* $\theta_\alpha = J_\beta\theta_{\alpha,\gamma} = -J_\gamma\theta_{\alpha,\beta}.$

Proof. The first formula follows directly from the system (2.18). Solving the system (2.18) with respect to C_α we obtain

$$(n^3-1)J_\beta C_\alpha = (\theta_\alpha - J_\gamma\theta_{\beta,\alpha}) + n(\theta_\beta - J_\alpha\theta_{\gamma,\beta}) + n^2(\theta_\gamma - J_\beta\theta_{\alpha,\gamma}). \tag{2.22}$$

Then (2.21) is a consequence of (2.22) and (2.20). The last assertion follows from (2.20) . **Q.E.D.**

Corollary 2.5 *On a 4n-dimensional ($n > 1$) QKT manifold the $sp(1)$-connection 1-forms are given by*

$$\omega_\beta = \frac{1}{2}J_\beta\left(\theta_\gamma - \theta_\beta + \frac{1}{1-n}\theta_\alpha\right) + \frac{1}{2(1-n)}\theta_{\alpha,\gamma}. \tag{2.23}$$

Proof. The proof follows in a straightforward way from (2.19), (2.20), (2.6) and (2.16). **Q.E.D.**

Theorem 2.2 and the above formulas lead to the following criterion

Proposition 2.6 *Let $(M, g, (H))$ be a 4n-dimensional ($n > 1$) QKT manifold. The following conditions are equivalent:*

i) $(M, g, (H))$ is a HKT manifold;

ii) $d_\alpha F^+_\alpha = d_\beta F^+_\beta = d_\gamma F^+_\gamma$;

iii) $\theta_\alpha = J_\beta\theta_{\gamma,\alpha}$.

Proof. If $(M, g, (H))$ is a HKT manifold, the connection 1-forms $\omega_\alpha = 0, \alpha = 1, 2, 3$. Then ii) and iii) follow from (2.16), (2.20), (2.9) and (2.8).

If iii) holds, then (2.20) and (2.19) yield $C_\alpha = A_\alpha = 0, \alpha = 1, 2, 3$, since $n > 1$. Consequently, $2\omega_\alpha = J_\beta C_\beta - J_\beta A_\beta = 0$ by (2.16) and (2.6). Thus the equivalence of i) and iii) is proved.

Let ii) holds. Then we compute that $\theta_\alpha = J_\gamma\theta_{\beta,\alpha}$. Since $n > 1$, the equality (2.22) leads to $C_\alpha = 0, \alpha = 1, 2, 3$, which forces $\omega_\alpha = 0, \alpha = 1, 2, 3$ as above. This completes the proof. **Q.E.D.**

The next theorem shows that QKT manifolds are stable under a conformal transformations.

Theorem 2.7 *Let $(M, g, (J_\alpha), \nabla)$ be a 4n-dimensional QKT manifold. Then every Riemannian metric $\bar{g}$ in the conformal class [g] admits a QKT connection. If $\bar{g} = fg$ for a positive function f then the QKT connection $\bar{\nabla}$ corresponding to $\bar{g}$ is given by*

$$\begin{aligned}\bar{g}(\bar{\nabla}_X Y, Z) = fg(\nabla_X Y, Z) &+ \frac{1}{2}\left(df(X)g(Y,Z) + df(Y)g(X,Z) - df(Z)g(X,Y)\right) \\ &+ \frac{1}{2}\left(J_\alpha df \wedge F_\alpha + J_\beta df \wedge F_\beta + J_\gamma df \wedge F_\gamma\right)(X,Y,Z).\end{aligned} \tag{2.24}$$

If M is compact then there exists a unique (up to homotety) metric $g_G \in [g]$ with co-closed torsion 1-form.

Proof. First we assume $n > 1$. We shall apply Theorem 2.2 to the quaternionic Hermitian manifold $(M, \bar{g} = fg, (J_\alpha) \in Q)$. We denote the objects corresponding to the metric $\bar{g}$ by a line above the symbol e.g. $\bar{F}_\alpha$ denotes the Kähler form of J_α with respect to $\bar{g}$. An easy calculation gives the following sequence of formulas

$$d_\alpha \bar{F}^+_\alpha = J_\alpha df \wedge F_\alpha + f d_\alpha F^+_\alpha; \quad \bar{\theta}_\alpha = \theta_\alpha + (2n-1) d \ln f; \quad \bar{\theta}_{\alpha,\gamma} = \theta_{\alpha,\gamma} - J_\beta d \ln f. \tag{2.25}$$

We substitute (2.25) into (2.9), (2.19) and (2.23) to get

$$\bar{K}_\alpha = K_\alpha - 2J_\beta d \ln f, \quad \bar{A} = A, \quad \bar{\omega}_\alpha = \omega_\alpha - J_\beta d \ln f. \tag{2.26}$$

Using (2.25) and (2.26) we verify that the conditions (2.8) with respect to the metric $\bar{g}$ are fulfilled. Theorem 2.2 implies that there exists a QKT connection $\bar{\nabla}$ with respect to $(\bar{g}, Q)$. Using the well known relation between the Levi-Civita connections of conformally equivalent metrics, (2.25) and (2.26), we obtain (2.24) from (2.10).

If $n = 1$ we define the new QKT connection with respect to $(\bar{g}, Q)$ by (2.24).

Using (2.24), we find that the torsion tensors T and $\bar{T}$ of ∇ and $\bar{\nabla}$ are related by

$$\bar{T} = fT + J_\alpha df \wedge F_\alpha + J_\beta df \wedge F_\beta + J_\gamma df \wedge F_\gamma. \tag{2.27}$$

Consequently, we obtain from (2.27) for the torsion 1-forms t and $\bar{t}$ that

$$\bar{t} = t - (2n+1) d \ln f. \tag{2.28}$$

If M is compact, we may apply to (2.28) the theorem of Gauduchon for the existence of a Gauduchon metric on a compact Weyl manifold [15, 16] to obtain the desired metric g_G. **Q.E.D.**

We shall call the unique metric with co-closed torsion 1-form on a compact QKT manifold the *Gauduchon metric.*

Corollary 2.8 *On a compact QKT manifold with closed (non exact) torsion 1-form the Gauduchon metric g_G cannot have positive definite Riemannian Ricci tensor. In particular, if it is an Einstein manifold then it is of non-positive scalar curvature.*

Further, if the Gauduchon metric is Ricci flat then the corresponding torsion 1-form t_G is parallel with respect to the Levi-Civita connection of g_G.

Proof. The two form dt is invariant under conformal transformations by (2.28). Then the Gauduchon metric has harmonic torsion 1-form i.e $dt = \delta t = 0$. The claim follows from the Weitzenböeck formula (see e.g. [8]) $\int_M \{|dt|^2 + |\delta t|^2\} \, dV = \int_M \{|\nabla^g t|^2 + Ric^g(t^\#, t^\#)\} \, dV = 0$, where $t^\#$ is the dual vector field of t, $|.|$ is the usual tensor norm and dV is the volume form. **Q.E.D.**

Theorem 2.7 allows us to supply a large class of (compact) QKT manifold. Namely, any conformal metric of a QK, HK or HKT manifold will give a QKT manifold. This leads to the notion of *locally conformally QK (resp. locally conformally HK, resp. locally conformally HKT) manifolds* (briefly l.c.QK (resp. l.c.HK, resp. l.c.HKT) manifolds) in the context of QKT geometry.

The l.c.QK and l.c.HK manifolds have already appeared in the context of Hermitian-Einstein-Weyl structures [36] and of 3-Sasakian structures [12]. These two classes of quaternionic manifolds are studied in detail (mostly in the compact case) in [34, 35].

We recall that a quaternionic Hermitian manifold (M, g, Q) is said to be l.c.QK (resp. l.c.HK, resp. l.c.HKT) manifold if each point $p \in M$ has a neighbourhood U_p such that $g\Big|_{U_p}$ is conformally equivalent to a QK (resp.HK, resp.HKT) metric. There are compact l.c.QK manifold which do not

admit any QK structure [34]. Typical examples of compact l.c. QK manifolds without any QK structure are the quaternionic Hopf spaces $H = (\mathcal{H}^n - \{0\})/\Gamma$, where Γ is an appropriate discrete group acting diagonally on the quaternionic coordinates in $\mathcal{H}^n$ (see [34]).

We recall that on a l.c.QK manifold the 4-form $\Omega = \sum_{\alpha=1}^{3} F_\alpha \wedge F_\alpha$ satisfies $d\Omega = \omega \wedge \Omega, d\omega = 0$, where ω is locally defined by $\omega = 2d\ln f$. On a l.c.QK manifold viewed as a QKT manifold by Theorem 2.7 the torsion 1-form is equal to $t = (2n+1)\omega$ by (2.28). The QK manifolds are Einstein provided the dimension is at least 8 [1, 7]. Then, the Gauduchon Theorem [16] applied to l.c.QK manifold in [34] can be stated in our context as follows

Corollary 2.9 *Let (M, g) be a compact $4n$-dimensional ($n > 1$) QKT manifold which is l.c.QK and assume that no metric in the conformal class $[g]$ of g is QK. Then the torsion 1-form of the Gauduchon metric g_G is parallel with respect to the Levi-Civita connection of g_G.*

Theorem 2.7, Theorem 2.2 together with Proposition 2.3 and Proposition 2.6 imply the following

Corollary 2.10 *Every l.c.QK manifold admits a QKT structure.*

Further, if $(M, g, (J_\alpha), \nabla)$ is a $4n$-dimensional $n > 1$ QKT manifold then:

i) $(M, g, (J_\alpha), \nabla)$ is a l.c.QK manifold if and only if

$$T = \frac{1}{2n+1}\left(t_\alpha \wedge F_\alpha + t_\beta \wedge F_\beta + t_\gamma \wedge F_\gamma\right), \quad dt = 0; \tag{2.29}$$

ii) $(M, g, (J_\alpha), \nabla)$ is a l.c.HKT manifold if and only if the 1-form $\theta_\alpha - J_\beta\theta_{\alpha,\gamma}$ is closed i.e.

$$d(\theta_\alpha - J_\beta\theta_{\alpha,\gamma}) = 0;$$

iii) $(M, g, (J_\alpha), \nabla)$ is a l.c.HK manifold if an only if (2.29) holds and

$$\theta_\alpha - J_\beta\theta_{\alpha,\gamma} = \frac{2(1-n)}{2n+1}t.$$

3 Curvature of a QKT space

Let $R = [\nabla, \nabla] - \nabla_{[,]}$ be the curvature tensor of type (1,3) of ∇. We denote the curvature tensor of type (0,4) $R(X, Y, Z, V) = g(R(X, Y)Z, V)$ by the same letter. There are three Ricci forms given by

$$\rho_\alpha(X, Y) = \frac{1}{2}\sum_{i=1}^{4n} R(X, Y, e_i, J_\alpha e_i), \quad \alpha = 1, 2, 3.$$

Proposition 3.1 *The curvature of a QKT manifold $(M, g, (J_\alpha), \nabla)$ satisfies the following relations*

$$R(X, Y)J_\alpha = \frac{1}{n}\left(\rho_\gamma(X, Y)J_\beta - \rho_\beta(X, Y)J_\gamma\right), \tag{3.30}$$

$$\rho_\alpha = d\omega_\alpha + \omega_\beta \wedge \omega_\gamma. \tag{3.31}$$

Proof. We follow the classical scheme (see e.g. [3, 25, 8]). Using (2.1) we obtain

$$R(X, Y)J_\alpha = -(d\omega_\beta + \omega_\gamma \wedge \omega_\alpha)(X, Y)J_\gamma + (d\omega_\gamma + \omega_\alpha \wedge \omega_\beta)(X, Y)J_\beta.$$

Taking the trace in the last equality, we get

$$\begin{aligned}\rho_\alpha(X,Y) &= \frac{1}{2}\sum_{i=1}^{4n} R(X,Y,e_i,J_\alpha e_i) = \frac{1}{2}\sum_{i=1}^{4n} R(X,Y,J_\beta e_i,J_\gamma e_i)\\ &= -\frac{1}{2}\sum_{i=1}^{4n} R(X,Y,e_i,J_\alpha e_i) + 2n(d\omega_\alpha + \omega_\beta\wedge\omega_\gamma)(X,Y)J_\beta.\end{aligned}$$

Q.E.D.

Using Proposition 3.1 we find a simple necessary and sufficient condition a QKT manifold to be a HKT one, i.e. the holonomy group of ∇ to be a subgroup of Sp(n).

Proposition 3.2 *A 4n-dimensional* ($n > 1$) *QKT manifold is a HKT manifold if and only if all the three Ricci forms vanish, i.e* $\rho_1 = \rho_2 = \rho_3 = 0$.

Proof. If a QKT manifold is a HKT manifold then the holonomy group of ∇ is contained in Sp(n). This implies $\rho_\alpha = 0, \quad \alpha = 1,2,3$.

For the converse, let the three Ricci forms vanish. The equations (3.31) mean that the curvature of the Sp(1) connection on Q vanish. Then there exists a basis $(I_\alpha, \alpha = 1,2,3)$ of almost complex structures on Q and each I_α is ∇-parallel i.e. the corresponding connection 1-forms $\omega_{I_\alpha} = 0, \alpha = 1,2,3$. Then each I_α is a complex structure, by (2.5) and (2.6). This implies that the QKT manifold is a HKT manifold. **Q.E.D.**

We denote by Ric, Ric^g the Ricci tensors of the QKT connection and of the Levi-Civita connection, respectively. In fact $Ric(X,Y) = \sum_{i=1}^{4n} R(e_i,X,Y,e_i)$.

Our main technical result is the following

Proposition 3.3 *Let* $(M,g,(J_\alpha),\nabla)$ *be a 4n-dimensional QKT manifold. The following formulas hold*

$$\begin{aligned}& n\rho_\alpha(X,J_\alpha Y) + \rho_\beta(X,J_\beta Y) + \rho_\gamma(X,J_\gamma Y) =\\ & -nRic(XY) + \frac{n}{4}(dT)_\alpha(X,J_\alpha Y) + \frac{n}{2}(\nabla T)_\alpha(X,J_\alpha Y);\end{aligned} \tag{3.32}$$

$$(n-1)\rho_\alpha(X,J_\alpha Y) = -\frac{n(n-1)}{n+2}Ric(X,Y) \tag{3.33}$$

$$+\frac{n}{4(n+2)}\left\{(n+1)(dT)_\alpha(X,J_\alpha Y) - (dT)_\beta(X,J_\beta Y) - (dT)_\gamma(X,J_\gamma Y)\right\}$$

$$+\frac{n}{2(n+2)}\left\{(n+1)(\nabla T)_\alpha(X,J_\alpha Y) - (\nabla T)_\beta(X,J_\beta Y) - (\nabla T)_\gamma(X,J_\gamma Y)\right\}, \tag{3.34}$$

where

$(dT)_\alpha(X,Y) = \sum_{i=1}^{4n} dT(X,Y,e_i,J_\alpha e_i), \quad (\nabla T)_\alpha(X,Y) = \sum_{i=1}^{4n}(\nabla_X T)(Y,e_i,J_\alpha e_i)$.

Proof. Since the torsion is a 3-form, we have

$$(\nabla^g_X T)(Y,Z,U) = (\nabla_X T)(Y,Z,U) + \frac{1}{2}\underset{XYZ}{\sigma}\left\{g(T(X,Y),T(Z,U)\right\}, \tag{3.35}$$

where $\underset{XYZ}{\sigma}$ denote the cyclic sum of X,Y,Z.

The exterior derivative dT is given by

$$dT(X,Y,Z,U) = \underset{XYZ}{\sigma}\{(\nabla_X T)(Y,Z,U) + g(T(X,Y),T(Z,U)\} - (\nabla_U T)(X,Y,Z) + \underset{XYZ}{\sigma}\{g(T(X,Y),T(Z,U)\}. \tag{3.36}$$

The first Bianchi identity for ∇ states

$$\underset{XYZ}{\sigma} R(X,Y,Z,U) = \underset{XYZ}{\sigma}\{(\nabla_X T)(Y,Z,U) + g(T(X,Y),T(Z,U)\}. \tag{3.37}$$

We denote by B the Bianchi projector i.e. $B(X,Y,Z,U) = \underset{XYZ}{\sigma} R(X,Y,Z,U)$.

The curvature R^g of the Levi-Civita connection is connected by R in the following way

$$\begin{aligned} R^g(X,Y,Z,U) &= R(X,Y,Z,U) - \frac{1}{2}(\nabla_X T)Y,Z,U) + \frac{1}{2}(\nabla_Y T)X,Z,U) \\ &- \frac{1}{2}g(T(X,Y),T(Z,U)) - \frac{1}{4}g(T(Y,Z),T(X,U)) - \frac{1}{4}g(T(Z,X),T(Y,U)). \end{aligned} \tag{3.38}$$

Define D by $D(X,Y,Z,U) = R(X,Y,Z,U) - R(Z,U,X,Y)$, we obtain from (3.38)

$$\begin{aligned} &D(X,Y,Z,U) = \\ &\frac{1}{2}(\nabla_X T)(Y,Z,U) - \frac{1}{2}(\nabla_Y T)(X,Z,U) - \frac{1}{2}(\nabla_Z T)(U,X,Y) + \frac{1}{2}(\nabla_U T)(Z,X,Y), \end{aligned} \tag{3.39}$$

since D^g of R^g is zero.

Using (3.30) and (3.37) we find the following relation between the Ricci tensor and the Ricci forms

$$\begin{aligned} \rho_\alpha(X,Y) &= -\frac{1}{2}\sum_{i=1}^{4n}(R(Y,e_i,X,J_\alpha e_i) + R(e_i,X,Y,J_\alpha e_i)) + \frac{1}{2}\sum_{i=1}^{4n} B(X,Y,e_i,J_\alpha e_i) \\ &= -\frac{1}{2}Ric(Y,J_\alpha X) + \frac{1}{2}Ric(X,J_\alpha Y) + \frac{1}{2}\sum_{i=1}^{4n} B(X,Y,e_i,J_\alpha e_i) \\ &+ \frac{1}{2n}\{\rho_\beta(J_\gamma Y,X) - \rho_\beta(J_\gamma X,Y) + \rho_\gamma(J_\beta X,Y) - \rho_\gamma(J_\beta Y,X)\}. \end{aligned} \tag{3.40}$$

On the other hand, using (3.30), we calculate

$$\begin{aligned} \sum_{i=1}^{4} D(X,e_i,J_\alpha e_i,Y) &= \sum_{i=1}^{4n}\{R(X,e_i,J_\alpha e_i,Y) + R(Y,e_i,J_\alpha e_i X)\} \\ &= -Ric(Y,J_\alpha X) - Ric(X,J_\alpha Y) \\ &+ \frac{1}{n}\{\rho_\beta(X,J_\gamma Y) + \rho_\beta(Y,J_\gamma X) - \rho_\gamma(Y,J_\beta X) - \rho_\gamma(X,J_\beta Y)\}. \end{aligned} \tag{3.41}$$

Combining (3.40) and (3.41), we derive

$$\begin{aligned} &n\rho_\alpha(X,J_\alpha Y) + \rho_\beta(X,J_\beta Y) + \rho_\gamma(X,J_\gamma Y) = \\ &-nRic(XY) + \frac{n}{2}B_\alpha(X,J_\alpha Y) + \frac{n}{2}D_\alpha(X,J_\alpha Y), \end{aligned} \tag{3.42}$$

where the tensors B_α and D_α are defined by $B_\alpha(X,Y) = \sum_{i=1}^{4n} B(X,Y,e_i,J_\alpha e_i)$ and $D_\alpha(X,Y) = \sum_{i=1}^{4n} D(X,e_i,J_\alpha e_i,Y)$. Taking into account (3.39), we get the expression

$$D_\alpha(X,Y) = \frac{1}{2}\sum_{i=1}^{4n}(\nabla_X T)(Y,e_i,J_\alpha e_i) + \frac{1}{2}\sum_{i=1}^{4n}(\nabla_Y T)(X,e_i,J_\alpha e_i) \quad \alpha = 1,2,3. \tag{3.43}$$

To calculate $B_\alpha + D_\alpha$ we use (3.36) twice and (3.43). After some calculations, we derive

$$B_\alpha(X,Y) + D_\alpha(X,Y) = \frac{1}{2}\sum_{i=1}^{4n} dT(X,Y,e_i,J_\alpha e_i) + \sum_{i=1}^{4n}(\nabla_X T)(Y,e_i,J_\alpha e_i), \quad \alpha = 1,2,3. \tag{3.44}$$

We substitute (3.44) into (3.42). Solving the obtained system, we obtain

$$\begin{aligned}(n-1)\left\{\rho_\alpha(X,J_\alpha Y) - \rho_\beta(X,J_\beta Y)\right\} &= \\ \frac{n}{2}\left\{(dT)_\alpha(X,J_\alpha Y) - (dT)_\beta(X,J_\beta Y)\right\} &+ \frac{n}{2}\left\{(\nabla T)_\alpha(X,J_\alpha Y) - (\nabla T)_\beta(X,J_\beta Y)\right\}.\end{aligned} \tag{3.45}$$

Finally, (3.42) and (3.45) imply (3.33). **Q.E.D.**

Remark 3. The Ricci tensor of a QKT connection is not symmetric in general. From (3.37), (3.35) and the fact that T is a 3-form we get the formula $Ric(X,Y) - Ric(Y,X) = \sum_{i=1}^{4n}(\nabla^g_{e_i}T)(e_i,X,Y) = -\delta T(X,Y)$. Hence, the Ricci tensor of a linear connection with totally skew-symmetric torsion is symmetric if and only if the torsion 3-form is co-closed.

4 QKT manifolds with parallel torsion and homogeneous QKT structures

Let $(G/K,g)$ be a reductive (locally) homogeneous Riemannian manifold. The canonical connection ∇ is characterised by the properties $\nabla g = \nabla T = \nabla R = 0$ [26],p.193. A homogeneous quaternionic Hermitian manifold (resp. homogeneous hyper Hermitian) manifold $(G/K,g,Q)$ is a homogeneous Riemannian manifold with an invariant quaternionic Hermitian subbundle Q (resp. three invariant anti commuting complex structures). This means that the bundle Q (resp. each of the three complex structures) is parallel with respect to the canonical connection ∇. The torsion of ∇ is totally skew-symmetric if and only if the homogeneous Riemannian manifold is naturally reductive [26] (see also [44, 33]. Homogeneous QKT (resp. HKT) manifolds are homogeneous quaternionic Hermitian (resp. homogeneous hyper Hermitian) manifold which are naturally reductive. Examples of homogeneous HKT and QKT manifolds are presented in [33]. The homogeneous QKT manifolds in [33] are constructed from homogeneous HKT manifolds.

In this section we generalise the result of [33] which states that there are no homogeneous QKT manifold with torsion 4-form dT of type (2,2) in dimensions greater than four. First, we prove the following technical result

Proposition 4.1 *Let $(M,g,(J_\alpha),\nabla)$ be a $4n$-dimensional $(n>1)$ QKT manifold with 4-form dT of type (2,2) with respect to each $J_\alpha, \alpha = 1,2,3$. Suppose that the torsion is parallel with respect to the QKT-connection. Then the Ricci forms ρ_α are given by*

$$\rho_\alpha(X,J_\alpha Z) = \lambda g(X,Y), \quad \alpha = 1,2,3, \tag{4.46}$$

where λ is a smooth function on M.

Proof. Let the torsion be parallel i.e. $\nabla T = 0$. Remark 3 shows that the Ricci tensor is symmetric. The equalities (3.36) and (3.37) imply

$$B(X,Y,Z,U) = \underset{XYZ}{\sigma} \{g(T(X,Y),T(Z,U)\} = \frac{1}{2} dT(X,Y,Z,U). \tag{4.47}$$

We get $D = 0$ from (3.39).

Suppose now that the 4-form dT is of type (2,2) with respect to each $J_\alpha, \alpha = 1,2,3.$. Then it satisfies the equalities

$$dT(X,Y,Z,U) = dT(J_\alpha X, J_\alpha Y, Z, U) + dT(J_\alpha X, Y, J_\alpha Z, U) + dT(X, J_\alpha Y, J_\alpha Z, U). \tag{4.48}$$

The similar arguments as we used in the proof of Proposition 2.1 but applying (4.48) instead of (2.4), yield

Lemma 4.2 *On a QKT manifold with 4-form dT of type (2,2) with respect to each $J_\alpha, \alpha = 1,2,3$, the following equalities hold:*

$$(dT)_1(X, J_1 Y) = (dT)_2(X, J_2 Y) = (dT)_3(X, J_3 Y), \tag{4.49}$$

$$(dT)_\alpha(X, J_\alpha Y) = -(dT)_\alpha(J_\alpha X, Y), \quad \alpha = 1,2,3. \tag{4.50}$$

We substitute (4.49), (4.47) and $D = 0$ into (3.45) and (3.33) to get

$$\rho_1(X, J_1 Y) = \rho_2(X, J_2 Y) = \rho_3(X, J_3 Y), \tag{4.51}$$

$$\rho_\alpha(X, J_\alpha Y) = -\frac{n}{n+2} Ric(X,Y) + \frac{n}{4(n+2)} (dT)_\alpha(X, J_\alpha Y), \quad \alpha = 1,2,3. \tag{4.52}$$

The equality (4.50) shows that the 2-form dT_α is a (1,1)-form with respect to J_α. Hence, the dT_α is (1,1)-form with respect to each $J_\alpha, \alpha = 1,2,3$, because of (4.49). Since the Ricci tensor Ric is symmetric, (4.52) shows that the Ricci tensor Ric is of hybrid type with respect to each J_α i.e. $Ric(J_\alpha X, J_\alpha Y) = Ric(X,Y), \alpha = 1,2,3$ and the Ricci forms $\rho_\alpha, \alpha = 1,2,3$ are (1,1)-forms with respect to all $J_\alpha, \alpha = 1,2,3$. Taking into account (3.30), we obtain

$$\begin{aligned} & R(X, J_\alpha X, Z, J_\alpha Z) + R(X, J_\alpha X, J_\beta Z, J_\gamma Z) \\ + \ & R(J_\beta X, J_\gamma X, Z, J_\alpha Z) + R(J_\beta X, J_\gamma X, J_\beta Z, J_\gamma Z) \\ = \ & \frac{1}{n} \left(\rho_\alpha(X, J_\alpha X) + \rho_\alpha(J_\beta X, J_\gamma X)\right) g(Z,Z) = \frac{2}{n} \rho_\alpha(X, J_\alpha X) g(Z,Z), \end{aligned} \tag{4.53}$$

where the last equality of (4.53) is a consequence of the following identity

$$\rho_\alpha(J_\beta X, J_\gamma X) = -\rho_\beta(J_\beta X, X) = \rho_\alpha(X, J_\alpha X).$$

The left side of (4.53) is symmetric with respect to the vectors X, Z because $D = 0$. Hence, $\rho_\alpha(X, J_\alpha X) g(Z,Z) = \rho_\alpha(Z, J_\alpha Z) g(X,X), \alpha = 1,2,3$. The last equality together with (4.51) implies (4.46). **Q.E.D.**

Theorem 4.3 *Let $(M, g, (J_\alpha))$ be a 4n-dimensional ($n > 1$) QKT manifold with 4-form dT of type (2,2) with respect to each $J_\alpha, \alpha = 1,2,3$. Suppose that the torsion is parallel with respect to the QKT-connection. Then $(M, g, (J_\alpha))$ is either a HKT manifold with parallel torsion or a QK manifold.*

Proof. We apply Proposition 4.1. If the function $\lambda = 0$ then $\rho_\alpha = 0, \alpha = 1,2,3,$ by (4.46) and Proposition 3.2 implies that the QKT manifold is actually a HKT manifold.

Let $\lambda \neq 0$. The condition (4.46) determines the torsion completely. We proceed involving (3.31) into the computations as in [24]. We calculate using (2.1) and (4.46) that

$$(\nabla_Z \rho_\alpha)(X,Y) = \lambda \{\omega_\beta(Z) F_\gamma(X,Y) - \omega_\gamma(Z) F_\beta(X,Y)\} - d\lambda(Z) F_\alpha(X,Y). \tag{4.54}$$

Applying the operator d to (3.30), we get taking into account (4.46) that

$$d\rho_\alpha = \lambda(F_\beta \wedge \omega_\gamma - \omega_\beta \wedge F_\gamma) \tag{4.55}$$

On the other hand, we have

$$d\rho_\alpha = \underset{XYZ}{\sigma} \{(\nabla_Z \rho_\alpha)(X,Y) + \lambda(T(X,Y,J_\alpha Z)\}, \quad \alpha = 1,2,3. \tag{4.56}$$

Comparing the left-hand sides of (4.55) and (4.56) and using (4.54), we derive

$$\lambda \underset{XYZ}{\sigma} \{(T(X,Y), J_\alpha Z)\} = d\lambda \wedge F_\alpha(X,Y,Z), \quad \alpha = 1,2,3.$$

The last equality implies $\lambda T = J_\alpha d\lambda \wedge F_\alpha, \quad \alpha = 1,2,3$. If λ is a non zero constant then $T = 0$ and we recover the result of [24]. If λ is not a constant then there exists a point $p \in M$ and a neighbourhood V_p of p such that $\lambda\Big|_{V_p} \neq 0$. Then

$$T = J_\alpha d \ln \lambda \wedge F_\alpha, \quad \alpha = 1,2,3. \tag{4.57}$$

We take the trace in (4.57) to obtain

$$4(n-1) J_\alpha d \ln \lambda = 0, \quad \alpha = 1,2,3. \tag{4.58}$$

The equation (4.58) forces $d\lambda = 0$ since $n > 1$ and consequently $T = 0$ by (4.57). Hence, the QKT space is a QK manifold which completes the proof. **Q.E.D.**

On a locally homogeneous QKT manifold the torsion and curvature are parallel and Theorem 4.3 leads to the following

Theorem 4.4 *A (locally) homogeneous 4n-dimensional* $(n > 1)$ *QKT manifold with torsion 4-form dT of type (2,2) is either (locally) homogeneous HKT space or a (locally) symmetric QK space.*

Theorem 4.4 shows that there are no homogeneous (proper) QKT manifolds with torsion 4-form of type (2,2) in dimensions greater than four which is proved in [33] by different methods using the Lie algebra arguments.

5 Four dimensional QKT manifolds

In dimension 4 the situation is completely different from that described in Theorem 2.2 and Theorem 4.3 in higher dimensions. For a given quaternionic structure on a 4-dimensional manifold $(M, g(H))$ (or equivalently, given an orientation and a conformal class of Riemannian metrics [19]) there are many QKT structures [24]. More precisely, all QKT structures associated with $(g, (H))$ depend on a 1-form ψ due to the general identity

$$*\psi = -J\psi \wedge F, \tag{5.59}$$

where $*$ is the Hodge $*$-operator, J is an g-orthogonal almost complex structure with Kähler form F (see [19]). Indeed, for any given 1-form ψ we may define a QKT-connection ∇ as follows: $\nabla = \nabla^g + \frac{1}{2} * \psi$. Conversely, any 3-form T can be represented by $T = - * (*T)$ and the connection given above is a quaternionic connection with torsion $T = *\psi$. Hence, a QKT structure on a 4-dimensional oriented manifold is a pair (g, t) of a Riemannian metric g and an 1-form t. The choice of g generates three almost complex structures $(J_\alpha), \alpha = 1, 2, 3$, satisfying the quaternionic identities [19]. The torsion 3-form T is given by

$$T = *t = t_\alpha \wedge F_\alpha = t_\beta \wedge F_\beta = t_\gamma \wedge F_\gamma. \tag{5.60}$$

As consequence of (5.59), we obtain $*dT = *d * t = -\delta t$. The last identity means that the torsion 3-form T is closed if and only if the 1-form t is co-closed. Thus, in dimension 4 there are many strong QKT structures.

In higher dimensions the conformal change of the metric induces a unique QKT structure by Theorem 2.7. We may define a QKT connection corresponding to a conformally equivalent metric $\bar{g} = fg$ in dimension 4 by (2.24) and call this conformal QKT transformation. In the compact case, taking the Gauduchon metric of Theorem 2.7, we obtain

Proposition 5.1 *Let $(M, g, (H), \nabla)$ be a compact 4-dimensional QKT manifold. In the conformal class [g] there exists a unique (up to homotety) strong QKT structure conformally equivalent to the given one.*

Further, we consider QKT structures with parallel torsion. We have

Theorem 5.2 *A 4-dimensional QKT manifold M with parallel torsion 3-form is a strong QKT manifold, the torsion 1-form is parallel with respect to the Levi-Civita connection and M is locally isometric to the product $N^3 \times \mathcal{R}$, where N^3 is a three dimensional Riemannian manifold admitting a Riemannian connection ∇ with totally skew-symmetric torsion, parallel with respect to ∇.*

Proof. The proof is based on the following

Lemma 5.3 *A 4-dimensional QKT manifold has parallel torsion 3-form if and only if it has parallel torsion 1-form with respect to the Levi-Civita connection.*

Proof of Lemma 5.3. We calculate using (5.60) and (2.1) that

$$\begin{aligned} (\nabla_Z T)(X, Y, U) &= t_\alpha(U) \left(\omega_\beta(Z) F_\gamma(Y, X) - \omega_\gamma(Z) F_\beta(Y, X)\right) \\ &- t_\alpha(X) \left(\omega_\beta(Z) F_\gamma(Y, U) - \omega_\gamma(Z) F_\beta(Y, U)\right) \\ &+ t_\alpha(Y) \left(\omega_\beta(Z) F_\gamma(X, U) - \omega_\gamma(Z) F_\beta(X, U)\right) \\ &+ F_\alpha(Y, U)(\nabla_Z t_\alpha)X + F_\alpha(X, Y)(\nabla_Z t_\alpha)U - F_\alpha(X, U)(\nabla_Z t_\alpha)Y. \end{aligned} \tag{5.61}$$

Taking the trace in (5.61), we obtain

$$\sum_{i=1}^{4} (\nabla_Z T)(X, e_i, J_\alpha e_i) = -2(\nabla_Z t_\alpha)X - 2\left(\omega_\beta(Z) t_\gamma(X) - \omega_\gamma(Z) t_\beta(X)\right). \tag{5.62}$$

Using (2.1), we get

$$(\nabla_Z t_\alpha)X = (\nabla_Z t) J_\alpha X - \left(\omega_\beta(Z) t_\gamma(X) - \omega_\gamma(Z) t_\beta(X)\right). \tag{5.63}$$

The equation (5.63) and (5.62) yield

$$\sum_{i=1}^{4}(\nabla_Z T)(J_\alpha X, e_i, J_\alpha e_i) = 2(\nabla_Z t)X, \quad \alpha = 1,2,3. \tag{5.64}$$

Then $\nabla t = 0$ since the torsion is parallel. But $\nabla^g t = \nabla t$ by (2.11) and (5.60). Hence, $\nabla^g t = 0$.

For the converse, we insert (5.63) into (5.61) to get

$$(\nabla_Z T)(X,Y,U) = F_\alpha(Y,U)(\nabla_Z t)J_\alpha X + F_\alpha(X,Y)(\nabla_Z t)J_\alpha U + F_\alpha(U,X)(\nabla_Z t)J_\alpha Y, \tag{5.65}$$

since the dimension is equal to four. If $\nabla^g t = 0$ then $\nabla t = 0$ and (5.65) leads to $\nabla T = 0$ which proves the lemma. **Q.E.D.**

Lemma 5.3 shows that (M,g) is locally isometric to the Riemannian product $\mathcal{R} \times N^3$ of a real line and a 3-dimensional manifold N^3 (see e.g. [26]). Using (5.60) we see that $T(t^\#, X^\perp, Y^\perp) = 0$ for every vector fields $X^\perp, Y^\perp$ orthonormal to the vector field $t^\#$ dual to the torsion 1-form t. Hence, the torsion T and therefore the connection ∇ descend to N^3.

In particular, $\delta t = 0$ and therefore the QKT structure is strong. **Q.E.D.**

As a consequence of Theorem 5.2, we recover the following two results proved in [27] in the setting of naturally reductive homogeneous 4-manifolds

Theorem 5.4 *A (locally) homogeneous 4-dimensional QKT manifold is locally isometric to the Riemannian product $\mathcal{R} \times N^3$ of a real line and a naturally reductive homogeneous 3-manifold N^3.*

Theorem 5.5 *Let (M,g) be a 4-dimensional compact homogeneous QKT manifold. Then the universal covering space $\tilde{M}$ of M is isometric to the Riemannian product $\mathcal{R} \times N^3$ of a real line and the three dimensional space N^3 is one of the following*

i) $R^3, S^3, \mathcal{H}^3$;

ii) isometric to one of the following Lie groups with a suitable left invariant metric:

1. $SU(2)$;

2. $\widetilde{SL(2,\mathcal{R})}$, the universal covering of $SL(2,\mathcal{R})$;

3. the Heisenberg group.

Theorem 5.5 is based on the classification of 3-dimensional simply connected naturally reductive homogeneous spaces given in [44].

5.1 Einstein-like QKT 4-manifolds

It is well known [7, 1] that a 4n-dimensional ($n > 1$) QK manifold is Einstein and the Ricci forms satisfy $\rho_\alpha(X, J_\alpha Y) = \rho_\beta(X, J_\beta Y) = \rho_\gamma(X, J_\gamma Y) = \lambda g(X,Y)$, where λ is a constant. However, the assumptions that these properties hold on a QKT manifold ($n > 1$) force the torsion to be zero [24] and the QKT manifold is a QK manifold. Actually, we have already generalised this result proving that if λ is not a constant the torsion has to be zero (see the proof of Theorem 4.3).

If the dimension is equal to 4 the situation is different. In this section we show that there exists a 4-dimensional (proper) QKT manifold satisfying similar curvature properties as those mentioned above.

We denote by K the following (0,2) tensor

$K(X,Y) := \rho_\alpha(X, J_\alpha Y) + \rho_\beta(X, J_\beta Y) + \rho_\gamma(X, J_\gamma Y)$.

The tensor K is independent of the chosen local almost complex structures (J_α) because of the following

Proposition 5.6 *Let $(M, g, (J_\alpha), \nabla)$ be a 4-dimensional QKT manifold. Then:*

$$K = -Ric + \nabla^g t - \frac{\delta t}{2} g; \tag{5.66}$$

$$Skew(Ric) = -\frac{1}{4} < dt, F_\alpha > F_\alpha + \frac{1}{2}(d^c t)_\alpha, \quad \alpha = 1, 2, 3; \tag{5.67}$$

$$Ric^g = Sym(Ric) + \frac{1}{2}(|t|^2 g - t \otimes t), \tag{5.68}$$

where $<,>$ is the scalar product of tensors induced by g, Skew (resp. Sym) denotes the skew-symmetric (resp. symmetric) part of a tensor.

In particular, the Ricci tensor is symmetric if and only if the torsion 1-form is closed.

Proof. We use (3.42). From (5.64) and (3.43), we obtain

$$D_\alpha(X, J_\alpha Y) = (\nabla_X t)Y - (\nabla_{J_\alpha Y} t)J_\alpha X, \quad \alpha = 1, 2, 3. \tag{5.69}$$

To compute B_α we need the following general identity

Lemma 5.7 *On a 4-dimensional QKT manifold we have $\underset{XYZ}{\sigma} g(T(X,Y), T(Z,U)) = 0$.*

Proof of Lemma 5.7. Since $\underset{XYZ}{\sigma} g(T(X,Y), T(Z,U))$ is a 4-form it is sufficient to check the equality for a basis of type $\{X, J_\alpha X, J_\beta X, J_\gamma X\}$. The last claim is obvious because of (5.60).

For each $\alpha \in \{1, 2, 3\}$, Lemma 5.7, (5.65) and (5.64) yield

$$B_\alpha(X, J_\alpha Y) = \sum_{i=1}^{4} \underset{XJ_\alpha Y e_i}{\sigma} (\nabla_X T)(J_\alpha Y, e_i, J_\alpha e_i) = (\nabla_X t)Y + (\nabla_{J_\alpha Y} t)J_\alpha X - \delta t g(X, Y). \tag{5.70}$$

Substituting (5.69), (5.70) into (3.42) and putting $n = 1$, we derive (5.66) since $\nabla^g t = \nabla t$. Taking the trace in (5.65), we get $\sum_{i=1}^4 (\nabla_{e_i} T)(e_i, X, Y) = \frac{1}{2} \sum_{i=1}^4 dt(e_i, J_\alpha e_i) F_\alpha(X,Y) + dt(J_\alpha X, J_\alpha Y), \alpha = 1, 2, 3$. Then (5.67) follows from the last equality and Remark 3. The equation (5.68) is a direct consequence of (3.38) and (5.60). **Q.E.D.**

A $4n$-dimensional QKT manifold $(M, g, (J_\alpha), \nabla)$ is said to be a *Einstein QKT manifold* if the symmetric part $Sym(Ric)$ of the Ricci tensor of ∇ is a scalar multiple of the metric g i.e. $Sym(Ric) = \frac{Scal}{4n} g$, where $Scal = tr_g Ric$ is the scalar curvature of ∇.

We note that the scalar curvature $Scal$ of an Einstein QKT manifold may not be a constant.

We shall say that a 4-dimensional QKT manifold is *sp(1)-Einstein* if the symmetric part $Sym(K)$ of the tensor K is a scalar multiple of the metric g since the tensor K is determined by the sp(1)-part of the curvature. On a sp(1)-Einstein QKT manifold $Sym(K) = \frac{Scal^K}{4} g$, where $Scal^K = tr_g K$.

For a given QKT manifold with torsion 1-form t we consider the corresponding Weyl structure ∇^W, i.e. the unique torsion-free linear connection determined by the condition

$$\nabla^W g = -t \otimes g. \tag{5.71}$$

Conversely, in dimension 4, to a given Weyl structure $\nabla^W g = \psi \otimes g$ we associate the QKT connection with torsion $T = *(-\psi)$. Note that a given Weyl structure on a conformal manifold (M,[g]) does not depend on the particularly chosen metric $g \in [g]$ but depends on the conformal class [g]. A Weyl structure is said to be *Einstein-Weyl* if the symmetric part $Sym(Ric^W)$ of its Ricci tensor is a scalar multiple of the metric g. Weyl structures and especially Einstein-Weyl structures have been much studied. For a nice overview of Einstein-Weyl geometry see [13]. The next theorem shows the link between Einstein-Weyl geometry and sp(1)-Einstein QKT manifolds in dimension 4.

Theorem 5.8 *Let $(M, g, (J_\alpha), \nabla)$ be a 4-dimensional QKT manifold with torsion 1-form t. The following conditions are equivalent:*

i) $(M, g, (J_\alpha), \nabla)$ is a sp(1)-Einstein QKT manifold.

ii) The corresponding Weyl structure is an Einstein-Weyl structure.

Proof. The Weyl connection ∇^W determined by (5.71) is given explicitly by

$$\nabla^W_X Y = \nabla^g_X Y + \frac{1}{2}t(X)Y + \frac{1}{2}t(Y)X - \frac{1}{2}g(X,Y)t^{\#}.$$

The symmetric part of its Ricci tensor is equal to

$$Sym(Ric^W) = Ric^g - Sym(\nabla^g t) - \frac{1}{2}(|t|^2 g - t \otimes t) + \frac{\delta t}{2} g. \tag{5.72}$$

Keeping in mind that $\nabla^g t = \nabla t$, we get from (5.66), (5.68) and (5.72) that $Sym(Ric^W) = -Sym(K)$. The theorem follows from the last equality. **Q.E.D.**

It is well known [6, 42] that on a 4-dimensional conformal manifold there exists a hypercomplex structure iff the conformal structure has anti-self-dual Weyl tensor (see also [19]). Every 4-dimensional hypercomplex manifold $(M, g, (H_\alpha))$, i.e. (an oriented anti-self-dual 4-manifold) carries a unique HKT structure in view of the results in [19, 17]. Indeed, let $\theta = \theta_\alpha = \theta_\beta = \theta_\gamma$ be the common Lee form. The unique HKT structure is defined by $\nabla = \nabla^g - \frac{1}{2} * \theta$ [19] (the uniqueness is a consequence of a general result in [17], see also [20]). The HKT structure on a 4-dimensional hypercomplex manifold is sp(1)-Einstein since the tensor K vanishes. The corresponding Weyl structure to the given HKT structure on a 4-dimensional hyperhermitian manifold is the Obata connection [19], i.e. the unique torsion-free linear connection which preserves each of the three hypercomplex structures. As a consequence of Theorem 5.8, we recover the result in [38] which states that the Obata connection of a hyper complex 4-manifold is Einstein-Weyl and the symmetric part of its Ricci tensor is zero.

Theorem 5.8 and (5.66) show that every Einstein-Weyl structure determined by (5.71) on a 4-dimensional conformal manifold whose vector field dual to the 1-form t is Killing, induces an Einstein and sp(1)-Einstein QKT structure.

Corollary 5.9 *Let $(M, [g], \nabla^W)$ be a compact 4-dimensional Einstein-Weyl manifold. Then the corresponding QKT structure to the Gauduchon metric of ∇^W is Einstein and sp(1)-Einstein.*

Proof. On a compact Einstein-Weyl manifold the vector field dual to the Lee form of the Gauduchon metric is Killing by the result of Tod [43]. Then the claim follows from Theorem 5.8 and (5.66). **Q.E.D.**

The Ricci tensor of a 4-dimensional QKT manifold is symmetric iff the torsion 1-form is closed by Proposition 5.6. Applying Theorem 3 in [16] and using Theorem 5.8, we obtain

Corollary 5.10 *Let $(M, g, (J_\alpha), \nabla)$ be a 4-dimensional compact sp(1)-Einstein QKT manifold with symmetric Ricci tensor. Suppose that the torsion 1-form is not exact. Then the torsion 1-form corresponding to the Gauduchon metric g_G of $(M, g, (J_\alpha), \nabla)$ is parallel with respect to the Levi-Civita connection of g_G and the universal cover of (M, g_G) is isometric to $\mathcal{R} \times S^3$. In particular, the quaternionic bundle (J_α) admits hypercomplex structure.*

A lot is known about Einstein-Weyl manifolds (see a nice survey [13]). There are many (compact) Einstein-Weyl 4-manifolds (e.g. $S^2 \otimes S^2$). Among them there are (anti)-self-dual as well as non (anti)-self-dual. We mention here the Einstein-Weyl examples of Bianchi IX type metric

[11, 28, 29]. All these Einstein-Weyl 4-manifolds admit sp(1)-Einstein QKT structures by Theorem 5.8.

It is also known that there are obstructions to the existence of Einstein-Weyl structures on compact 4-manifold [37]. If the manifold M is finitely covered by $T^2 \otimes S^2$ which cannot be Einstein-Weyl then M does not admit Einstein-Weyl structure and therefore there are no sp(1)-Einstein structures on M.

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